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MODAL LOGICS WITH SEVERAL OPERATORS AND PROVABILITY INTERPRETATIONS

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ABSTRACT
The results herein solve positively some conjectures of Smoryński by generalizing results of Solovay (1976). The proofs rest on a modification of the usual semantics for modal logic and Solovay's techniques.

§1. Introduction

M_n is the modal language consisting of

propositional variables:	$p_0, p_1, p_2, \ldots$
propositional constant:	$\perp$
logical connectives:	$\vee, \wedge, \neg, \rightarrow$
modal operators:	$\Box_1, \Box_2, \ldots, \Box_n$

The *formulas* of M_n are constructed inductively in the usual way so that $\Box_i A$ is a formula if A is. A *theory* in the language M_n is a collection of formulas of M_n which contains all tautologies and is closed under modus ponens. If $\mathscr{A}$ is a set of formulas of M_n and B is a formula of M_n then $\mathscr{A} \vdash B$ means B is in the theory generated by $\mathscr{A}$.

PA is the standard first order formalization of Peano Arithmetic (e.g. as in [4]). $\bar{0}$ is the constant representing 0 and for each natural number n let $\bar{n}$, the *numeral* for n, be $S^n\bar{0}$. For each formula φ in the language of PA, $\ulcorner\varphi\urcorner$ is the Gödel number of φ and if the free variables of φ are among $v_1, \ldots, v_k$ then $\ulcorner\varphi(\dot{x}_1, \ldots, \dot{x}_k)\urcorner$ is the function definable in PA which represents the substitution of

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the numerals for $x_1,\dots,x_k$ in the free occurrences of $v_1,\dots,v_k$ respectively. In particular,

$$\mathrm{PA}\vdash \ulcorner\varphi(\bar{n}_1,\dots,\bar{n}_k)\urcorner = \ulcorner\varphi(\dot{\bar{n}}_1,\dots,\dot{\bar{n}}_k)\urcorner.$$

All first order theories are tacitly assumed to be axiomatizable (i.e. recursively enumerable) and to be equipped with an interpretation of PA. If T is a first order theory then Pr_T is a unary predicate Σ_1-definable in PA which formalizes provability in T. In particular, Pr_T satisfies the following *derivability conditions*:

D1. $T\vdash\varphi$ iff $\mathrm{PA}\vdash\mathrm{Pr}_T(\ulcorner\varphi\urcorner)$,

D2. $\mathrm{PA}\vdash\mathrm{Pr}_T(\ulcorner\varphi\urcorner)\rightarrow\mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\varphi\urcorner)\urcorner)$,

D3. $\mathrm{PA}\vdash\mathrm{Pr}_T(\ulcorner\varphi\urcorner)\wedge\mathrm{Pr}_T(\ulcorner\varphi\rightarrow\psi\urcorner)\rightarrow\mathrm{Pr}_T(\ulcorner\psi\urcorner)$,

D4. $\mathrm{PA}\vdash\varphi(x_1,\dots,x_n)\rightarrow\mathrm{Pr}_T(\ulcorner\varphi(\dot{x}_1,\dots,\dot{x}_k)\urcorner)$ for Σ_1 formulas.

A theory T is *Σ_1-sound* if every Σ_1 arithmetic sentence proved by T is true; T is *arithmetically sound* if every arithmetic sentence proved by T is true.

Given theories $T_1,\dots,T_n$ an *interpretation of M_n* with respect to $T_1,\dots,T_n$ is an assignment $p\mapsto p^*$ of arithmetic sentences to propositional variables. $*$ is extended to all formulas of M_n inductively by preserving logical connectives and so that $\perp^*$ is $\bar{0}\neq\bar{0}$ and $(\Box_i A)^*$ is $\mathrm{Pr}_{T_i}(\ulcorner A^*\urcorner)$. A formula A of M_n is *T-valid* with respect to $T_1,\dots,T_n$ if $T\vdash A^*$ for all interpretations $*$; A is *valid* with respect to $T_1,\dots,T_n$ if A^* is true (in the standard model) for all $*$.

The primary purpose of this paper is to study the decidability of the collection of valid formulas for various $T_1,\dots,T_n$. Note that if the collection of formulas of M_n which are valid with respect to $T_1,\dots,T_n$ is decidable then the collection of T_i-valid formulas is decidable (remember that all first order theories are assumed to be Σ_1-sound). Generalizing the solution by Boolos [1] of a problem of Friedman [2], Solovay [6] showed that if $n=1$ then the collection of valid formulas is decidable. Smoryński conjectured that the set of valid formulas with respect to PA, ZF is decidable. The main result of this paper, Theorem 2 of Section 3, proves a generalization of Smoryński's conjecture. An axiomatization is given for the collection of valid formulas. The relevant modal theories and the appropriate semantics are developed in Section 2. The central modal theories, PRL(n), are generalizations of PRL(2) which Smoryński suggested (under a different name) and which in turn generalizes G of [6]. Section 4 contains what I consider an amusing application: There is a Boolean combination of Σ_1 formulas, φ, such that $\mathrm{PA}+\mathrm{Con(PA)}\vdash\mathrm{Con}(\mathrm{PA}+\varphi)$ and $\mathrm{PA}+\mathrm{Con(PA)}\vdash\mathrm{Con}(\mathrm{PA}+\neg\varphi)$ while $\mathrm{ZF}\nvdash\mathrm{Con}(\mathrm{PA}+\mathrm{Con(PA)}+\varphi)$ and $\mathrm{ZF}\nvdash\mathrm{Con}(\mathrm{PA}+\mathrm{Con(PA)}+\neg\varphi)$.

§2. The theory PRL_n

PRL_n is the theory in M_n generated from the axiom schemas

A1. Tautologies,

A2. $\Box_i A \wedge \Box_i (A \to B) \to \Box_i B$,

A3. $\Box_i A \to \Box_j \Box_i A$,

A4. $\Box_i (\Box_i A \to A) \to \Box_i A$,

using the rules of inference

R1. $\dfrac{\mathrm{A} \quad \mathrm{A} \to \mathrm{B}}{\mathrm{B}}$ (modus ponens)

R2. $\dfrac{\mathrm{A}}{\Box_i A}$

An *M_n-model* is a tuple $\mathcal{K} = (K, \lhd, \Vdash, D_1, \ldots, D_n)$ where $\lhd$ is a strict partial ordering of K, $\Vdash$ is a subset of $K \times \{p_n : n \in \omega\}$ and $D_i \subseteq K$ for $i = 1, \ldots, n$. K is called the *universe* of $\mathcal{K}$ and $\Vdash$ is the *forcing relation* of $\mathcal{K}$. The intuition (as with standard Kripke models) is that the elements of K are possible worlds, and for $w \in K$, $\{p : w \Vdash p\}$ consists of all propositions true in w. $\Vdash$ is extended inductively to other formulas, so that $w \nVdash \bot$ and

$$
\begin{array}{lll}
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B, \\
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B, \\
w \Vdash \neg A & \text{iff} & w \nVdash A, \\
w \Vdash A \to B & \text{iff} & w \nVdash A \text{ or } w \Vdash B, \\
w \Vdash \Box_i A & \text{iff} & u \Vdash A \text{ whenever } w \lhd u \in D_i;
\end{array}
$$

$\mathcal{K}$ is a *model* of A if $w \Vdash A$ for each $w \in K$. $\mathcal{K}$ is a *model* of a set of formulas $\mathcal{A}$ if $\mathcal{K}$ is a model of each element of $\mathcal{A}$. $\mathrm{Th}(\mathcal{K})$ is the collection of all A for which $\mathcal{K}$ is a model. If $\mathcal{K}' = (K', \lhd', \Vdash', D'_1, \ldots, D'_n)$ is an M_n-model then K' is a *submodel* of $\mathcal{K}$ provided $K' \subseteq K$ and $\lhd'$, $\Vdash'$, $D'_1, \ldots, D'_n$ are the restrictions of $\lhd$, $\Vdash$, $D_1, \ldots, D_n$ respectively to K'.

LEMMA 1. *If $\mathcal{K}$ is an M_n-model then* $\mathrm{Th}(\mathcal{K})$ contains A1–A3 *and is closed under* R1 *and* R2.

PROOF. Straightforward. ■

Assume $\mathcal{A}$ is a collection of formulas of M_n containing the schemas A1–A3 and let T be the theory in M_n generated from $\mathcal{A}$ using R1 and R2. The *canonical model* of T, $\mathcal{K}_T$, will be constructed intermittently with the following lemmas.

Let $\hat{K}$ be the set of all complete consistent theories in the language M_n which extend T. For $1 \leqq i \leqq n$ and $w \in \hat{K}$, $w/\Box_i$ is the set of formulas A such that $\Box_i A \in w$.

LEMMA 2. *$w/\Box_i$ is a theory extending T.*

PROOF. R2 guarantees that $w/\Box_i$ extends T, and $w/\Box_i$ is closed under modus ponens by A2. ■

For $1 \leqq i \leqq n$ define a binary relation $<_i$ on $\hat{K}$ by $w_1 <_i w_2$ iff $w_1/\Box_i \subseteq w_2$.

LEMMA 3. *If $w_1 <_j w_2 <_i w_3$ then $w_1 <_i w_3$. In particular, $<_i$ is transitive.*

PROOF. Assume $w_1 <_j w_2 <_i w_3$, i.e. $w_1/\Box_j \subseteq w_2$ and $w_2/\Box_i \subseteq w_3$. Suppose $\Box_i A \in w_1$. By A3 and modus ponens, $\Box_j \Box_i A \in w_1$. Therefore, $\Box_i A \in w_2$ and $A \in w_3$. ■

$\hat{\mathcal{K}}$ is the generalized Kripke model $(\hat{K}, <_1, \ldots, <_n, \Vdash\!\vdash)$ where $w \Vdash\!\vdash p$ iff $p \in w$. Extend $\Vdash\!\vdash$ to other formulas of M_n as before in the case of logical connectives and so that $w \Vdash\!\vdash \Box_i A$ iff $u \Vdash\!\vdash A$ whenever $w <_i u$.

LEMMA 4. *$W \Vdash\!\vdash A$ iff $A \in w$.*

PROOF. By induction on the complexity of A.

For A a proposition variable this is by definition, $A = \perp$ is clear, and the induction is obvious if A has one of the forms $B \vee C$, $B \wedge C$, $\neg B$ or $B \to C$.

Suppose $A = \Box_i B$.

Assume $A \notin w$. $B \notin w/\Box_i$ so $w/\Box_i \cup \{\neg B\}$ generates a consistent theory v (using only modus ponens) by Lemma 2. Extend v to a complete consistent theory u. $w <_i u$ and by the induction hypothesis $u \Vdash\!\vdash \neg B$. This implies $w \not\Vdash\!\vdash A$.

Now assume $A \in w$. $B \in w/\Box_i$ so if $w <_i u$ then $u \not\Vdash\!\vdash B$ by the induction hypothesis. Therefore $w \Vdash\!\vdash A$. ■

Let $<$ be the transitive closure of $<_1 \cup <_2 \cup \cdots \cup <_n$.

LEMMA 5. *If $w_1 < w_2 <_i w_3$ then $w_1 <_i w_3$.*

PROOF. By Lemma 3. ■

$\mathcal{K}_T$ is the M_n-model $(K, \subseteq, \Vdash, D_1, \ldots, D_n)$ where K consists of all finite sequences $\langle w_1, \ldots, w_k \rangle$ of elements of $\hat{K}$ with $w_1 < w_2 < \cdots < w_k$, $\langle w_1, \ldots, w_k \rangle \Vdash p$ iff $p \in w_k$ and D_i consists of all $\langle w_1, \ldots, w_{k+1} \rangle$ such that $w_k <_i w_{k+1}$.

LEMMA 6. *$\langle w_1, \ldots, w_k \rangle \Vdash A$ iff $A \in w_k$.*

PROOF. By induction on the complexity of A.

As in Lemma 4, the nontrivial case is $A = \Box_i B$.

Assume $A \notin w_k$. $w_k \nVvdash \Box_i B$ so $w_1 \Vvdash \neg B$ for some $w \in \hat{K}$ with $w_k <_i w$. $\langle w_1, \ldots, w_k, w \rangle \in D_i$ and by Lemma 4 and the induction hypothesis $\langle w_1, \ldots, w_k, w \rangle \Vdash \neg B$. Therefore, $\langle w_1, \ldots, w_k \rangle \nVdash A$.

Now suppose $A \in w_k$. If $\langle w_1, \ldots, w_k, w_{k+1}, \ldots, w_t \rangle \in D_i$ then $w_k <_i w_t$ by the previous lemma so $B \in w_t$ and $\langle w_1, \ldots, w_t \rangle \Vdash B$. Therefore $\langle w_1, \ldots, w_k \rangle \Vdash A$. ∎

LEMMA 7. *$\mathcal{K}_T$ is a model of T.*

PROOF. Immediate. ∎

LEMMA 8. *If $\Box_i A \to \Box_j A$ is in T for all A then $D_i \supseteq D_j$.*

PPROOF. The assumption implies $w/\Box_i \to w/\Box_j$ for all $w \in K^*$. Hence, if $w_1 <_j w_2$ then $w_1 <_i w_2$ which implies $D_i \supseteq D_j$. ∎

THEOREM 1. *Assume $\mathcal{A}$ is a set of formulas of M_n containing schemas* A1–A3 *and let T be the theory generated from $\mathcal{A}$ using* R1 *and* R2. *If B is a formula of M_n the following are equivalent*:

1. *$B \in T$.*
2. *Every model of T is a model of B.*
3. *$\mathcal{K}_T$ is a model of B.*

PROOF. ($1 \Rightarrow 2$) By Lemma 1.

($2 \Rightarrow 3$) By Lemma 7.

($3 \Rightarrow 1$) Suppose $B \notin T$. Let w be a complete consistent theory containing T and $\neg B$. In $\mathcal{K}_T$, $w \nVdash B$. ∎

REMARK. Assume $\mathcal{A}$ is a set of formulas of M_n containing A1–A3 with the property that $\Box_1 A, \ldots, \Box_n A \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and A doesn't have the form $\Box_i B$. If T is the theory generated by $\mathcal{A}$ using R1 and R2 then $B \in T$ iff $\mathcal{A} \vdash B$. This may be proved syntactically or by modifying the construction of $\mathcal{K}_T$ by letting $\hat{K}$ be all complete consistent theories containing $\mathcal{A}$.

This implies that for any $\mathcal{A}$ containing A1–A3, if B can be derived from $\mathcal{A}$ using R1 and R2 then B has a derivation in which all applications of R2 come before any application of R1.

LEMMA 9. *Assume T is a theory in M_n which contains* A1–A4 *and is closed under $R1$ and $R2$. If $\mathcal{A}$ is a finite set of formulas of M_n, $\mathcal{K} = (K, \lhd, \Vdash, D_1, \ldots, D_n)$ is a model of T and $w_1, \ldots, w_k \in K$ then there is a finite submodel $\mathcal{K}' = (K', \lhd', \Vdash', D_1', \ldots, D_n')$ of $\mathcal{K}$ whose universe contains $w_1, \ldots, w_k$ such that for each w in the universe of $\mathcal{K}'$ and each $A \in \mathcal{A}$*

$$w \Vdash' A \quad \textit{iff} \quad w \Vdash A.$$

PROOF. Without loss of generality $\mathscr{A}$ is closed under subformulas. Let $A_1, \ldots, A_m$ list the elements of $\mathscr{A}$ of the form $\Box_i B$. If $1 \leqq j \leqq m$ and $A_j = \Box_i B$ choose $f_j : K \to K$ (a Skolem function for A_j) so that $f_j(w) = w$ if $w \Vdash A_j$ and if $w \Vdash \neg A_j$ then $w \lhd f_j(w) \in D_i$ and $f_j(w) \Vdash \neg B \wedge \Box_i B$. This is possible by A4.

CLAIM 1. *If $K' \subseteq K$ is closed under $f_1, \ldots, f_m$ and $\mathscr{K}'$ is the submodel of $\mathscr{K}$ with universe K' then $w \Vdash' A$ iff $w \Vdash A$ whenever $w \in K'$ and $A \in \mathscr{A}$.*

The proof of Claim 1 is straightforward induction on the complexity of A.

Note that $f_j(w) \unlhd u$ implies $f_j(u) = u$. This easily implies

CLAIM 2. *The closure of $\{w_1, \ldots, w_k\}$ under $f_1, \ldots, f_m$ is finite.*

By Claims 1 and 2 the lemma follows. ■

An M_n-model $\mathscr{K} = (K, \lhd, \Vdash, D_1, \ldots, D_n)$ is *treelike* if $(K, \lhd)$ is a tree, that is, $\{u \in K : u \lhd w\}$ is linearly ordered for $w \in K$.

Assume $\mathscr{P}$ is a strict partial ordering of $\{1, \ldots, n\}$. PRL($\mathscr{P}$) is the theory in the language M_n generated by R1 and R2 from A1–A4 along with the schema

$$\mathrm{A}\mathscr{P}.\ \Box_i A \to \Box_j A \qquad \text{for } i <_{\mathscr{P}} j.$$

LEMMA 10. *If $\mathscr{K} = (K, \lhd, \Vdash, D_1, \ldots, D_n)$ is a finite M_n-model with $D_i \supseteq D_j$ whenever $i <_{\mathscr{P}} j$ then $\mathscr{K}$ is a model of* PRL($\mathscr{P}$).

PROOF. To check A4 use the fact that K is finite (in fact the assumption that $\lhd$ has no infinite increasing chain $w_0 \lhd w_1 \lhd w_2 \lhd \cdots$ is enough). ■

THEOREM 2. *Assume A is a formula of M_n and T is the theory generated by* PRL($\mathscr{P}$) $\cup \{A\}$ *where $\mathscr{P}$ is a strict partial ordering of $\{1, \ldots, n\}$. If B is a formula of M_n then $B \in T$ iff every finite treelike model of A which satisfies $D_i \supseteq D_j$ whenever $i <_{\mathscr{P}} j$ is a model of B.*

PROOF. ($\Rightarrow$) By Theorem 1 and Lemma 10.

($\Leftarrow$) Suppose $B \in T$. There is $w \in K_T$ such that $w \Vdash \neg B$ (in $\mathscr{K}_T$) by Theorem 1. By Lemma 9 there's a finite submodel $\mathscr{K}'$ of $\mathscr{K}_T$ which is a model of A but not of B. $\mathscr{K}'$ is treelike since $\mathscr{K}_T$ is. By Lemma 8, $D'_i \supseteq D'_j$ if $i <_{\mathscr{P}} j$ where $\mathscr{K}' = (K', \subseteq, \Vdash', D'_1, \ldots, D'_n)$. ■

COROLLARY. PRL($\mathscr{P}$) is recursive.

§3. Decidability of the collection of valid formulas

If T is a first order theory then Refl(T) is the schema of sentences of the form $\mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi$.

The main result of this section is

THEOREM 1. *Assume $T_1, \ldots, T_n$ are first order theories which are arithmetically sound and T_j contains* $\mathrm{Refl}(T_i)$ *if $i<j$. The collection of formulas which are valid with respect to $T_1, \ldots, T_n$ is recursive.*

Note that the conclusion of the theorem implies that the set of T_i-valid formulas is recursive since A is T_i-valid with respect to $T_1, \ldots, T_n$ iff $\Box_i A$ is valid.

If $\mathscr{P}$ is the usual linear ordering of $\{1, \ldots, n\}$ write $\mathrm{PRL}(n)$ for $\mathrm{PRL}(\mathscr{P})$. $\mathrm{PRL}(n)^+$ is the theory in M_n generated by R1 and R2 from $\mathrm{PRL}(n)$ along with formulas of the form $\Box_j(\Box_i B \to B)$ where $i<j$. Refl_i is the schema consisting of the formulas of the form $\Box_i B \to B$.

Theorem 1 follows from

THEOREM 2. *Assume $T_1, \ldots, T_n$ are arithmetically sound first order theories such that T_j contains* $\mathrm{Refl}(T_i)$ *for $i<j$. If A is a formula of M_n*

(1) *the following are equivalent*:

(a) *A is valid with respect to $T_1, \ldots, T_n$,*

(b) $\mathrm{PRL}(n)^+ + \mathrm{Refl}_i (1 \leqq i \leqq n) \vdash A$,

(c) *the following is a theorem of* $\mathrm{PRL}(n)$:

$$(\bigwedge\{\Box_j(\Box_i B \to B) : \Box_i B \in S(A) \text{ and } i<j\}) \wedge (\bigwedge\{\Box_i B \to B : \Box_i B \in S(A)\}) \to A;$$

(2) *the following are equivalent for $1 \leqq k \leqq n$*:

(a) *A is T_k-valid with respect to $T_1, \ldots, T_n$,*

(b) $\mathrm{PRL}(n)^+ + \mathrm{Refl}_i \ (1 \leqq i < k) \vdash A$,

(c) *the following is a theorem of* $\mathrm{PRL}(n)$:

$$\begin{aligned}&(\bigwedge\{\Box_j(\Box_i B \to B) : \Box_i B \in S(A) \text{ and } i<j\})\\ &\quad \wedge (\bigwedge\{\Box_i B \to B : \Box_i B \in S(A) \text{ and } i<k\}) \to A;\end{aligned}$$

where $S(A)$ is the collection of subformulas of A.

The proof will use a modification of the function h from [6].

Suppose $\mathscr{K} = (K, \lhd, \Vdash, D_1, \ldots, D_n)$ is a finite treelike M_n-model where $K = \{1, \ldots, m\}$ and 1 is the smallest element of K with respect to $\lhd$. Also assume $K = D_1 \supseteq D_2 \supseteq \cdots \supseteq D_n$. Fix k with $1 \leqq k \leqq n$.

Let $\mathscr{K}_k = (\{0, 1, \ldots, m\}, \lhd, \Vdash^k, D_1^k, \ldots, D_n^k)$ where we extend $\lhd$ to 0 so that $0 \lhd 1$, $\Vdash^k$ extends $\Vdash$ and $0 \Vdash^k p$ iff $1 \Vdash p$, and

$$D_i^k = \begin{cases} D_i \cup \{1\} & \text{if } i \leqq k, \\ D_i - \{1\} & \text{if } k < i. \end{cases}$$

Set $D^k_{n+1} = \varnothing$ for convenience. Note that for any formula A of M_n, $x \Vdash^k A$ iff $x \Vdash A$ for $x \neq 0$.

Using the recursion theorem, there is a Σ_1-definable function h_k in PA such that PA proves

$$h_k(0) = 0$$

$$h_k(a+1) = \begin{cases} x & \text{if } a \text{ is the Godel number of a proof of} \\ & \bar{x} \neq \ell_k \text{ in } T_i \text{ and } h_k(a) \lhd x \in D^k_i, \\ h_k(a) & \text{otherwise,} \end{cases}$$

where ℓ_k is a constant defined in PA to be $\lim_{a\to\infty} h_k(a)$. One can prove in PA that $h_k(a) \unlhd h_k(a+1)$ so that ℓ_k is defined.

LEMMA 1. $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{\mathrm{PA}}(\ulcorner \bar{x} \unlhd \ell_k \urcorner)$.

PROOF. $\mathrm{PA} \vdash y = h_k(a) \to \mathrm{Pr}_{\mathrm{PA}}(\ulcorner \dot{y} = h_k(\dot{a})\urcorner)$ *by using induction on a.* ■

LEMMA 2. *If* $x \neq 0$ *and* $x \notin D^k_{i+1}$ *then* $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{T_i}(\ulcorner \bar{x} \neq \ell_k \urcorner)$.

PROOF. Immediate from the choice of h_k. ■

LEMMA 3. $T_i \vdash \ell_k \neq \bar{0} \to \ell_k \in D^k_i$.

PROOF. This is obvious for $i = 1$ since $D^k_1 = K$.
Suppose $i \neq 1$, $x \neq 0$ and $x \notin D^k_i$. By the previous lemma

$$\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{T_{i-1}}(\ulcorner \bar{x} \neq \ell_k \urcorner).$$

Since $T_i \supseteq \mathrm{Refl}(T_{i-1})$,

$$T_i \vdash \mathrm{Pr}_{T_{i-1}}(\ulcorner \bar{x} \neq \ell_k \urcorner) \to \bar{x} \neq \ell_k.$$

Therefore $T_i \vdash \bar{x} \neq \ell_k$. ■

Define $\mathrm{Con}(T + \varphi)$ to be $\neg \mathrm{Pr}_T(\ulcorner \neg \varphi \urcorner)$.

LEMMA 4. *If* $x \lhd y \in D^k_i$ *then* $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Con}(T_i + \bar{y} = \ell_k)$.

PROOF. Immediate from the properties of h_k. ■

Define an interpretation $*$ with respect to $T_1, \ldots, T_n$ by letting p^* be $\bigvee \{\bar{x} = \ell_k : x \Vdash p\}$. Note that p^* is equivalent to a Boolean combination of Σ_1 sentences in PA. In fact, if p has the property that if $\{w \in K : w \Vdash p\}$ is closed upwards with respect to $\lhd$ then p^* is equivalent to a Σ_1 sentence in PA.

LEMMA 5. *Assume A is a formula of* M_n *such that* $1 \Vdash \Box_j(\Box_i B \to B)$ *whenever*

$\Box_i B \in S(A)$ with $i < j$ and $1 \Vdash \Box_i B \to B$ whenever $\Box_i B \in S(A)$ and $i < k$. If $x \neq 0$ then $x \Vdash A$ implies $\mathrm{PA} \vdash \bar{x} = \ell_k \to A^*$ and $x \Vdash \neg A$ implies $\mathrm{PA} \vdash \bar{x} = \ell_k \to \neg A^*$.

PROOF. By induction on the complexity of A.

If A is a propositional variable or constant the conclusion is immediate from the definition of $*$, and the inductive step for $\vee, \wedge, \neg$ or $\to$ is straightforward.

Suppose $A = \Box_i B$ and $0 \neq x \leqq m$.

Case 1. $x \Vdash A$

First suppose $x \in D^k_{i+1}$. Note that $x \Vdash B$. For if $x = 1$ then $i < k$ and $x \Vdash \Box_i B \to B$ and if $x \neq 1$ then $x \Vdash \Box_i B \to B$ since $1 \Vdash \Box_{i+1}(\Box_i B \to B)$. $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{T_i}(\ulcorner \bar{x} \trianglelefteq \ell_k \in D^k_i \urcorner)$ by Lemmas 1 and 3. If $x \trianglelefteq y \in D^k_i$ then $y \Vdash B$ and $\mathrm{PA} \vdash \bar{y} = \ell_k \to B^*$. Hence, $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{T_i}(\ulcorner B^* \urcorner)$ which is the desired conclusion since $(\Box_i B)^* = \mathrm{Pr}_{T_i}(\ulcorner B^* \urcorner)$.

Now suppose $x \notin D^k_{i+1}$. By Lemmas 1, 2 and 3 $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Pr}_{T_i}(\ulcorner \bar{x} \lhd \ell_k \in D^k_i \urcorner)$. $\mathrm{PA} \vdash A^*$ follow as above.

Case 2. $x \Vdash \neg A$

Choose $y \in D^k_i$ such that $x \lhd y$ and $y \Vdash \neg B$. By the induction hypothesis $\mathrm{PA} \vdash \bar{y} = \ell_k \to \neg B^*$. By Lemma 4, $\mathrm{PA} \vdash \bar{x} = \ell_k \to \mathrm{Con}(T_i + \neg B^*)$ which is the desired conclusion. ■

LEMMA 6. *In addition to the hypothesis of Lemma 5 assume $n = k$ and $1 \Vdash \Box_n B \to B$ for all $\Box_n B \in S(A)$:*

(1) $0 \Vdash A$ *iff* $1 \Vdash A$,

(2) $0 \Vdash A$ *implies* $\mathrm{PA} \vdash \bar{0} = \ell_k \to A^*$,

(3) $0 \Vdash \neg A$ *implies* $\mathrm{PA} \vdash \bar{0} = \ell_k \to \neg A^*$.

PROOF. (1) is checked by an easy induction on A.

(2) and (3) are also proved by induction on A using (1) and an argument similar to that used to prove Lemma 5. ■

LEMMA 7. *$\bar{0} = \ell_k$ is true (in the standard model).*

PROOF. If ℓ_k represents x in the standard model and $x \neq 0$ then $\mathrm{Pr}_{T_i}(\ulcorner \bar{x} \neq \ell_k \urcorner)$ is true in the standard model for some i. In that case $T_i \vdash \bar{x} \neq \ell_k$ and by Σ_1-soundness of T_i, $\bar{x} \neq \ell_k$ is true in the standard model — contradiction. ■

LEMMA 8. *$T_k + \bar{1} = \ell_k$ is consistent.*

PROOF. Since $1 \in D^k_k$, this follows from Lemmas 4 and 7 (and the fact that PA is true). ■

PROOF OF THEOREM 2. A1–A3, A$\mathscr{P}$ are clearly valid with respect to $T_1,\ldots,T_n$, A4 interprets as formalized instances of Löb's Theorem ([3]) which follow from PA, and the collection of formulas which are PA-valid with respect to $T_1,\ldots,T_n$ is closed under R1 and R2. Also, $\Box_j(\Box_i B \to B)$ is PA-valid with respect to $T_1,\ldots,T_n$ if $i<j$, and $\Box_i B \to B$ is T_j-valid with respect to $T_1,\ldots,T_n$ if $i<j$. By the assumption that each T_i is arithmetically sound, $\Box_i B \to B$ is valid with respect to $T_1,\ldots,T_n$. Hence (1)(b)$\Rightarrow$(1)(a) and (2)(b)$\Rightarrow$(2)(a).

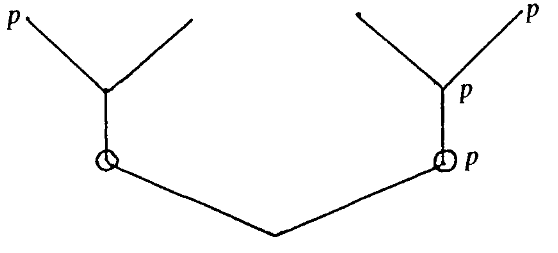
(2)(a)$\Rightarrow$(2)(c). Assume (2)(c) fails. By Theorem 2 of Section 2 let $\mathscr{K} = (K, \lhd, \Vdash, D_1,\ldots,D_n)$ be a finite treelike model of PRL(n) which is not a model of the formula, F, from (2)(c). Without loss of generality $K=\{1,\ldots,m\}$ and $1 \Vdash \neg F$. By reducing the universe of $\mathscr{K}$ if necessary, it may also be assumed that 1 is the smallest element of K with respect to $\lhd$ and $K = D$. Construct $\mathscr{K}_k$ and the interpretation $*$ as above. By Lemma 8, $T_k + \bar{1} = \ell_k$ is consistent. By Lemma 5 and the fact that $1 \Vdash \neg A$, $T_k + \bar{1} = \ell_k \vdash \neg A^*$. Hence $T_k \nvdash A^*$ and A is not T_k-valid with respect to $T_1,\ldots,T_n$.

(1)(a)$\Rightarrow$(1)(c). The argument is similar using Lemmas 6 and 7 in place of Lemmas 5 and 8.

REMARK. Arithmetic soundness of each T_i is necessary for (1)(b)$\Rightarrow$(1)(a), the assumption of arithmetic soundness may be dropped for (2)(b)$\Rightarrow$(2)(a) and both (1)(a)$\Rightarrow$(1)(c) and (2)(a)$\Rightarrow$(2)(c) follow from just Σ_1-soundness.

§4. An example

Consider the following M_2-model:



D_1 is the entire universe, D_2 is comprised of the circled nodes and the nodes labelled p are those which force p. Let A be the conjunction of the formulas

$$\neg\Box_2\neg\Box_1(\Box_1\bot \vee \neg p),$$
$$\neg\Box_2\neg\Box_1(\Box_1\bot \vee p),$$
$$\Box_1(\Box_1\bot \vee \neg\Box_1 p),$$
$$\Box_1(\Box_1\bot \vee \neg\Box_1\neg p).$$

One can check that the bottom node forces $\Box_2(\Box_1 B \to B)$ whenever $\Box_1 B \in S(A)$, $\Box_2 B \to B$ whenever $\Box_2 B \in S(A)$ and $\Box_1 B \to B$ whenever $\Box_1 B \in S(A)$. The bottom node also forces A. By Theorem 2, part 2 of Section 3 (or Lemma 5) there's an interpretation $*$ with respect to PA, ZF such that A^* is true and p^* is a Boolean combination of Σ_1 sentences. Let $T = \mathrm{PA} + \mathrm{Con(PA)}$. A^* is equivalent to

$$\mathrm{ZF} \nvdash \mathrm{Con}(T + p^*),$$

$$\mathrm{ZF} \nvdash \mathrm{Con}(T + \neg p^*),$$

$$T \vdash \mathrm{Con}(\mathrm{PA} + p^*),$$

$$T \vdash \mathrm{Con}(\mathrm{PA} + \neg p^*).$$

Notice that p^* is independent of ZF and cannot be chosen to be a Σ_1 sentence.

Acknowledgements

Theorem 2 of Section 3 is a generalization of conjectures made by C Smoryński. The results in this paper were obtained in the fall of 1982 while he was visiting Ohio State University. I would like to thank him for bringing his conjectures to my attention and for several enlightening conversations. I would also like to thank Robert Solovay for helpful correspondence. In particular, the characterization in part (2)(c) of Theorem 2 from Section 3 is a generalization of observations made by him and me.

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